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Modified Measurement Differentiation Method for Stochastic Control Systems

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This paper presents a modified measurement differentiation method for the stochastic control systems with both white and colored measurement noises. The first key point of this method is the use of a first-order shaping filter, i.e., a colored noise to approximate the white measurement noise. In order to preserve the wide-band nature of the white noise, the bandwidth of the shaping filter is set to be 10 times that of the system fastest mode. In addition, the root-mean-square value of the shaping filter output is set to be equal to that of the original white noise. The second key point involves augmenting this new colored noise with the original colored ones. The other processes are similar to the traditional measurement differentiation method. This way the proposed method can take both colored and white measurement noises into consideration without increasing the order of the state equations. However, this advantage cannot be obtained by the measurement differentiation method or the state augmentation method. A stochastic optimal control system with time-varying colored and white measurement noises is given for illustration.

Nomenclature

$A(t)[A_a(t)]$ = $n \times n$ time-varying system matrix of original (augmented) state equation
A_t = target acceleration
$a(t)$ = $m \times 1$ white-noise vector that generates colored noise $v(t)$
$a_1(t)$ = white noise that generates colored noise $v_1(t)$
$B(t)[B_a(t)]$ = $n \times r$ time-varying input coefficient matrix of original (augmented) state equation
B_w = bandwidth of shaping filter, which is equal to $10w_0$
$b(t)$ = $p \times 1$ colored measurement noise vector
$b_1(t)$ = colored noise to replace white noise $d_w(t)$
$C(t)$ = $q \times n$ time-varying output coefficient matrix
$C_1(t)$ = $k \times q$ cross-variance matrix of $w(t)$ and $v_{\text{white}}(t)$
$D(t), d_{ij}$ = $q \times m$ time-varying gain matrix of $v(t)$ with entries d_{ij}
$D_a(t)$ = $q \times (m+p)$ time-varying gain matrix of $v_a(t)$
$d(t)$ = $p \times 1$ white measurement noise vector
$d_1(t)$ = scalar gain of $v_1(t)$
$d_w(t)$ = white fading measurement noise
$E(t), e_{ij}$ = $q \times p$ time-varying gain matrix of $d(t)$ with entries e_{ij}
$e_1(t)$ = scalar gain of $d_w(t)$
$F(t)[F_a(t)]$ = $n \times k$ time-varying gain coefficient matrix of $w(t)$ for original (augmented) state equation
$f(t)$ = $p \times 1$ white-noise vector that generates $b(t)$
$f_1(t)$ = white noise that generates $b_1(t)$
$G(t)$ = $m \times m$ bandwidth coefficient matrix of $v(t)$
g = bandwidth of colored receiver noise $v(t)$
g_1 = bandwidth of colored noise $v_1(t)$
H_1 = $q \times n$ newly defined time-varying output matrix
h_1 = $\dot{d}_1 + d_1 g_1$
h_2 = $\dot{e}_1 - 10 e_1 w_0$
$J(t)$ = $p \times p$ covariance matrix of $d(t)$
$J_1(t)$ = variance of $d_w(t)$
$J_f(t)$ = $p \times p$ covariance matrix of $f(t)$ that is equal to $2J(t)/B_w$ or $2J_1(t)/B_w$
$K(t)$ = $n \times q$ Kalman gain matrix
$K_1(t)$ = KL_1
$K_2(t)$ = KL_2
k = number of system white noises
k_1, k_2, k_3 = Kalman gains
$L(t)$ = $q \times (m+p)$ matrix ($= \dot{D}_a + D_a W_a$)
$L_1(t)$ = $q \times q$ matrix
$L_2(t)$ = $q \times q$ matrix
m = number of colored measurement noises
n = number of states
$N_a(t)$ = $(m+p) \times 1$ white-noise vector
$P(t), p_{ij}$ = $n \times n$ covariance matrix of $x - \hat{x}$ with entries p_{ij}
p = number of white measurement noises
$Q(t)$ = $k \times k$ covariance matrix of $w(t)$
q = number of measurement outputs
$R(t)$ = $m \times m$ covariance matrix of $a(t)$
$R_1(t)$ = variance of $a_1(t)$
$R_2(t)$ = variance of $w_g(t)$
R_0 = homing range
r = number of inputs
T = transposition of vector or matrix
t_f = total time of engagement for homing range R_0
$u(t)$ = $r \times 1$ input vector
V_c = closing velocity of missile and target
$v(t)$ = $m \times 1$ colored measurement noise vector
$v_a(t)$ = $(m+p) \times 1$ augmented colored measurement noise vector
$v_{\text{white}}(t)$ = $q \times 1$ equivalent white-noise vector
$v_1(t)$ = colored measurement noise
$v_2(t)$ = glint measurement noise
$W_a(t)$ = $(m+p) \times (m+p)$ gain matrix of $v_a(t)$
w_a = augmented state noise that is equal to $[w\ a]^T$
$w(t)$ = $k \times 1$ system white-noise vector
w_g = white noise that generates colored glint noise
w_t = white noise that generates target maneuver
w_0 = bandwidth of system fastest mode
$x(t)$ = $n \times 1$ original (augmented) state vector
$y(t)$ = $q \times 1$ measurement (output) vector
Y_d = lateral distance of missile and target
$z(t)$ = $q \times 1$ newly defined measurement equation
β = root-mean-square value of target acceleration, ft/s^2
Φ_g = power spectral density of glint noise
σ_{rms} = root-mean-square value of glint noise
$\Sigma(t)$ = $(m+p) \times (m+p)$ covariance matrix of $N_a(t)$
$\Sigma_1(t)$ = $q \times q$ covariance matrix of $v_{\text{white}}(t)$
ν = number of target acceleration crossing zero

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I. Introduction

In general the noise of stochastic control system is assumed to be white. However, the bandwidths of some physical noises are comparable to those of the control signals. Therefore, they should be treated as colored noises, for example, the glint noise and target maneuver of a missile guidance system.[1,2] This type of stochastic control problem is always solved by augmenting the states of colored noises into the original state equation,[3] and then only white noises remain in the measurement equation. This method is very simple, and it can be applied to time-varying systems. However, it will increase both the order of states and the estimation load. On the other hand, some measurement differentiation methods[4,5] are developed by differentiating the output equation and then defining a new measurement equation to make the resulting noises be white. The advantage of this method is that it does not increase the order of the state equation. However, it cannot be applied when there are both colored and white measurement noises. The other method is the combination of state augmentation and output differentiation.[6–9] The problem is solved by first augmenting the states of colored noises with the original state equation and making part of the resulting output noise free. Then the remaining processes are similar to those of the measurement differentiation method by defining a new measurement equation to make the resulting measurement noise white. In addition, a reduced-order observer can be obtained by this method. However, if the number of independent colored measurement noises is larger than that of the outputs, the final order of the augmented state observer will be larger than that of the original state equation.

In this paper a modified measurement differentiation method is proposed for time-varying stochastic control systems with both white and colored measurement noises. The main ideas of this method are as follows: First, we replace the white measurement noise by a wide-band colored one and model the colored noise by a first-order shaping filter.[2] In addition, the root-mean-square value of the shaping filter output is set to be equal to that of the original white noise. This assumption is reasonable, because the white noise does not exist in reality,[8] and the bandwidth of the physical system is always a finite number. Thus only those white-noise frequencies below the bandwidth of the system fastest mode can influence the system performance. Therefore, it is sufficient to set the bandwidth of the shaping filter to be 10 times that of the system fastest mode. Second, we augment this new colored noise with the original colored ones, and the remaining processes are similar to those of the traditional measurement differentiation method. Thus the proposed method can take both colored and white measurement noises into consideration without increasing the order of the state equation, whereas this feature cannot be obtained by the measurement differentiation method or the state augmentation method. The details are developed in Sec. II.

It should be noted that the proposed method cannot be applied when the number of measurement outputs is less than that of the augmented colored noise. The way to solve this problem is by increasing the number of outputs and then using the measurement differentiation method or state augmentation method.[3–9] A guided missile control system[10] with time-varying colored and white measurement noises is given for discussion and illustration in Sec. III. Finally, a conclusion is drawn in Sec. IV.

II. Proposed Method

Let the state and measurement equations of a time-varying stochastic control system be

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + F(t)w(t) \tag{1}$$

$$y(t) = C(t)x(t) + D(t)v(t) + E(t)d(t) \tag{2}$$

respectively, where

$x(t) = n \times 1$ state vector
$A(t) = n \times n$ time-varying system matrix
$B(t) = n \times r$ time-varying input coefficient matrix
$u(t) = r \times 1$ input vector
$F(t) = n \times k$ time-varying gain coefficient matrix
$w(t) = k \times 1$ system white-noise vector with zero mean and covariance matrix $Q(t)$
$y(t) = q \times 1$ measurement (or output) vector
$C(t) = q \times n$ time-varying output coefficient matrix
$D(t) = q \times m$ time-varying colored noise gain coefficient matrix
$E(t) = q \times p$ time-varying white-noise gain coefficient matrix
$d(t) = p \times 1$ white-noise vector with zero mean and covariance matrix $J(t)$

Let $v(t)$, the $m \times 1$ colored-noise vector, be defined as

$$\dot{v}(t) = G(t)v(t) + a(t) \tag{3}$$

where

$G(t) = m \times m$ bandwidth coefficient matrix of $v(t)$
$a(t) = m \times 1$ white-noise vector with zero mean and covariance matrix $R(t)$

It should be noted that all the aforementioned white noises are assumed to be uncorrelated with one another. The first step of the proposed method is to use a *first-order shaping filter*, i.e., a wide-band colored noise, to approximate the white measurement noise. This assumption is reasonable, because the white noise is a convenient mathematical abstraction; it never exists in reality, and the bandwidth of the physical system is always finite. Therefore, only those white-noise frequencies below the bandwidth of the system fastest mode can influence system performance. Thus, one can approximate and replace the white-noise vector $d(t)$ by a wide-band colored one $b(t)$. Note that it is sufficient to set the bandwidth of the shaping filter (B_w) to 10 times that of the system fastest mode w_0, i.e.,

$$d(t) \cong b(t) \tag{4}$$

and

$$\dot{b}(t) = -B_w b(t) + B_w f(t), \quad B_w = 10w_0 \tag{5}$$

where $f(t)$ is a $p \times 1$ white-noise vector with zero mean, and in order for $b(t)$ to preserve the root-mean-square value of the original white noise,[10] the covariance matrix $J_f(t)$ of $f(t)$ is set to be related to that of $d(t)$, i.e., $J(t)$, as $2J(t)/B_w$.

By Eqs. (2) and (4), one can rewrite Eq. (2) as

$$y(t) = C(t)x(t) + D_a(t)v_a(t) \tag{6}$$

where $D_a(t)v_a(t)$ is the augmented colored-noise term with

$$D_a(t) = [D(t) \quad E(t)]_{q\times(m+p)} \tag{7}$$

and

$$v_a(t) = [v(t) \quad b(t)]^T \tag{8}$$

where the superscript T denotes matrix or vector transposition. By Eqs. (3), (5), and (8), the augmented colored-noise vector $v_a(t)$ is defined as

$$\dot{v}_a(t) = \begin{bmatrix} \dot{v}(t) \\ \dot{b}(t) \end{bmatrix} = \begin{bmatrix} G(t) & 0 \\ 0 & -10w_0 I \end{bmatrix} \begin{bmatrix} v(t) \\ b(t) \end{bmatrix} + \begin{bmatrix} a(t) \\ 10w_0 f(t) \end{bmatrix} = W_a(t)v_a(t) + N_a(t) \tag{9}$$

where

$$W_a(t) = \begin{bmatrix} G(t) & 0 \\ 0 & -10w_0 I \end{bmatrix} \tag{10}$$

$$N_a(t) = \begin{bmatrix} a(t) \\ 10w_0 f(t) \end{bmatrix} \tag{11}$$

Therefore, $N_a(t)$ is a white-noise vector with zero mean and covariance matrix $\Sigma(t)$, where

$$\Sigma(t) = \begin{bmatrix} R(t) & 0 \\ 0 & 100w_0^2 J_f(t) \end{bmatrix} \tag{12}$$

The second step is similar to the original measurement differentiation method but with some extensions to time-varying systems, by defining a new measurement equation to make the resulting measurement noise processes white. There are three cases depending on the following $q \times (m+p)$ matrix:

$$L(t) = \dot{D}_a(t) + D_a(t)W_a(t) \tag{13}$$

Case 1. L is a square nonsingular matrix.
Case 2. L is a rectangular matrix $[q > (m+p)]$ with rank $m+p$.
Case 3. L is a square or rectangular zero matrix, i.e.,

$$L = 0_{q\times(m+p)} \tag{14}$$

It is proposed that the following new measurement equation can be used for any one of the three cases to make the resulting measurement noises white:

$$z = L_1\dot{y} - L_2 y - L_1 C B u \tag{15}$$

Note that for the sake of clarity the time-dependent notation in Eq. (15) and thereafter is omitted and that L_1 and L_2 are defined as follows:

Case 1

$$L_1 = D_a L^{-1} \tag{16}$$

$$L_2 = I_{q\times q} \tag{17}$$

Case 2

$$L_1 = D_a(L^T L)^{-1}L^T \tag{18}$$

$$L_2 = I_{q\times q} \tag{19}$$

Case 3

$$L_1 = I_{q\times q} \tag{20}$$

$$L_2 = 0_{q\times q} \tag{21}$$

By Eqs. (6), (9), and (15), one has

$$\begin{aligned} L_1\dot{y} &= L_1(\dot{C}x + C\dot{x} + \dot{D}_a v_a + D_a\dot{v}_a) \\ &= L_1[\dot{C}x + C(Ax + Bu + Fw) + \dot{D}_a v_a \\ &\quad + D_a(W_a v_a + N_a)] \\ &= L_1(\dot{C} + CA)x + L_1CBu + L_1CFw \\ &\quad + L_1(\dot{D}_a + D_aW_a)v_a + L_1D_aN_a \\ &= L_1(\dot{C} + CA)x + L_1CBu + L_1CFw \\ &\quad + L_1Lv_a + L_1D_aN_a \end{aligned} \tag{22}$$

For case 1, using Eqs. (6), (15–17), and (22), one has

$$\begin{aligned} z &= L_1\dot{y} - L_2y - L_1CBu \\ &= L_1(\dot{C} + CA)x + L_1CBu + L_1CFw + L_1Lv_a \\ &\quad + L_1D_aN_a - (Cx + D_av_a) - L_1CBu \\ &= [L_1(\dot{C} + CA) - C]x + L_1CFw + L_1D_aN_a \\ &= H_1x + v_{\text{white}} \end{aligned} \tag{23}$$

where

$$H_1 = L_1(\dot{C} + CA) - C \tag{24}$$

and v_{white} is the combination of the uncorrelated white noises w and N_a, i.e.,

$$v_{\text{white}} = L_1CFw + L_1D_aN_a \tag{25}$$

Therefore, v_{white} is white noise with zero mean and covariance Σ_1:

$$\Sigma_1 = L_1CFQ(L_1CF)^T + L_1D_a\Sigma(L_1D_a)^T \tag{26}$$

For case 2, using Eqs. (6), (15), (18), (19), and (22), one has

$$\begin{aligned} z &= L_1\dot{y} - L_2y - L_1CBu \\ &= L_1(\dot{C} + CA)x + L_1CBu + L_1CFw + D_a(L^TL)^{-1}L^TLv_a \\ &\quad + L_1D_aN_a - (Cx + D_av_a) - L_1CBu \\ &= [L_1(\dot{C} + CA) - C]x + L_1CFw + L_1D_aN_a \\ &= H_1x + v_{\text{white}} \end{aligned} \tag{27}$$

where

$$H_1 = L_1(\dot{C} + CA) - C \tag{28}$$

$$v_{\text{white}} = L_1CFw + L_1D_aN_a \tag{29}$$

Therefore, comparing Eqs. (23) and (27), one can see that the form of the new measurement equation (27) of case 2 is the same as that of case 1, whereas L_1 and L_2 are defined by Eqs. (18) and (19).

For case 3, using Eqs. (6), (14), (15), and (20–22), one has

$$\begin{aligned} z &= L_1\dot{y} - L_2y - L_1CBu \\ &= L_1(\dot{C} + CA)x + L_1CBu + L_1CFw + L_1D_aN_a - L_1CBu \\ &= [L_1(\dot{C} + CA)]x + L_1CFw + L_1D_aN_a \\ &= H_1x + v_{\text{white}} \end{aligned} \tag{30}$$

where

$$H_1 = L_1(\dot{C} + CA) \tag{31}$$

$$v_{\text{white}} = L_1CFw + L_1D_aN_a \tag{32}$$

Therefore, the form of the new measurement equation (30) of case 3 is also similar to Eq. (23) of case 1, whereas L_1 and L_2 are defined by Eqs. (20) and (21). The cross-variance matrix C_1 of the state noise w and the new measurement noise v_{white} for any one of cases 1, 2, and 3 is

$$C_1 = Q(L_1CF)^T \tag{33}$$

By the original state equation (1) and any of the newly defined measurement equations (24), (27), and (30), one has the Kalman gain matrix[4]

$$K = (PH_1^T + BC_1)\Sigma_1^{-1} \tag{34}$$

where P is the covariance matrix of the estimated state error defined as

$$\dot{P} = AP + PA^T + FQF^T - K\Sigma_1K^T \tag{35}$$

The estimated state equation is obtained as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(z - H_1\hat{x}) \\ &= A\hat{x} + Bu + K(L_1\dot{y} - L_2y - L_1CBu - H_1\hat{x}) \\ &= A\hat{x} + Bu + K_1\dot{y} - K_2y - K_1CBu - KH_1\hat{x} \end{aligned} \tag{36}$$

where

$$K_1 = KL_1 \tag{37}$$

$$K_2 = KL_2 \tag{38}$$

The derivative of the output $K_1\dot{y}$ in Eq. (36) can be avoided by the following substitution[4]:

$$K_1\dot{y} = \frac{d}{dt}(K_1y) - \dot{K}_1y \tag{39}$$

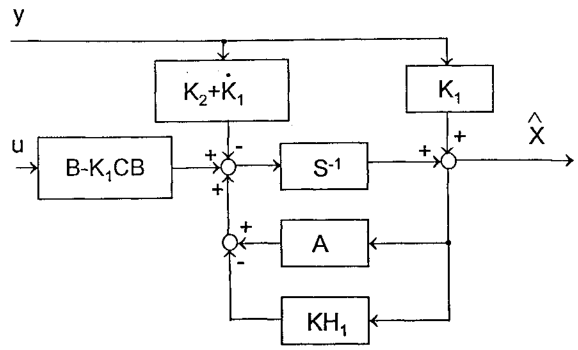


Fig. 1 Block diagram of Kalman filter for optimal control system with colored measurement noise.

From Eqs. (36) and (39) one can implement the estimated state equation as

$$\frac{d}{dt}(\hat{x} - K_1 y) = \dot{\hat{x}} - K_1 \dot{y} - \dot{K}_1 y$$

$$= A\hat{x} + Bu - (K_2 + \dot{K}_1)y$$

$$- K_1 C B u - K H_1 \hat{x} \qquad (40)$$

The block diagram of the Kalman filter for this control system is shown in Fig. 1. It should be noted that, by the proposed method of approximating the white noises by the colored ones and then augmenting these new colored noises to the original colored ones, the order of the state equation is not increased. However, it will be increased by the state augmentation method. In addition, this new method can take both colored and white measurement noises into consideration. Therefore, it is more general than the original measurement differentiation methods.

III. Discussions and Examples

It should be noted from Sec. II that the proposed method can be applied only for the conditions as

$$q \geq m + p \qquad \text{or} \qquad L = 0 \qquad (41)$$

Because if

$$q < m + p \qquad \text{and} \qquad L \neq 0 \qquad (42)$$

then any method of cases 1, 2, and 3 cannot be applied. This is because under this condition either L^{-1} in Eq. (16) or $L^T L$ in Eq. (18) is singular. This may be shown by the following example, with $q = 1, m = 1$ and $p = 1$; then, by Eq. (2), one has

$$y = Cx + d_1 v_1 + e_1 d_w \qquad (43)$$

where d_w is a white noise and v_1 is a colored noise, defined as

$$\dot{v}_1 = g_1 v_1 + a_1 \qquad (44)$$

where a_1 is a white noise with variance R_1. Then by Eqs. (4) and (5) one can approximate the white noise d_w by a colored noise b_1, i.e.,

$$d_w \cong b_1 \qquad (45)$$

and b_1 is defined as

$$\dot{b}_1 = -10 w_0 b_1 + 10 w_0 f_1 \qquad (46)$$

By Eqs. (43) and (44), one can augment this new colored noise with the original one as follows:

$$y = Cx + D_a v_a \qquad (47)$$

where

$$D_a = [d_1 \quad e_1] \qquad (48)$$

$$v_a = \begin{bmatrix} v_1 \\ b_1 \end{bmatrix} \qquad (49)$$

By Eqs. (44), (46), and (49) one has

$$\dot{v}_a = \begin{bmatrix} \dot{v}_1 \\ \dot{b}_1 \end{bmatrix}$$

$$= \begin{bmatrix} g_1 & 0 \\ 0 & -10w_0 \end{bmatrix} \begin{bmatrix} v_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_1 \\ 10 w_0 f_1 \end{bmatrix}$$

$$= W_a v_a + N_a \qquad (50)$$

where

$$W_a = \begin{bmatrix} g_1 & 0 \\ 0 & -10w_0 \end{bmatrix} \qquad (51)$$

$$N_a = \begin{bmatrix} a_1 \\ 10 w_0 f_1 \end{bmatrix} \qquad (52)$$

Therefore, by Eqs. (13), (48), and (51), one has

$$L = \dot{D}_a + D_a W_a$$

$$= [\dot{d}_1 \quad \dot{e}_1] + [d_1 \quad e_1] \begin{bmatrix} g_1 & 0 \\ 0 & -10w_0 \end{bmatrix}$$

$$= [\dot{d}_1 + d_1 g_1 \quad \dot{e}_1 - 10 e_1 w_0]$$

$$= [h_1 \quad h_2] \qquad (53)$$

where

$$h_1 = \dot{d}_1 + d_1 g_1 \qquad (54)$$

$$h_2 = \dot{e}_1 - 10 e_1 w_0 \qquad (55)$$

Therefore, it is quite clear that L^{-1} does not exist, and

$$\det(L^T L) = \det\left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} [h_1 \quad h_2] \right)$$

$$= \det \begin{bmatrix} h_1^2 & h_1 h_2 \\ h_2 h_1 & h_2^2 \end{bmatrix}$$

$$= 0 \qquad (56)$$

Thus both Eqs. (16) and (18) of cases 1 and 2 cannot be applied. In addition, since, by Eq. (42), $L \neq 0$, i.e., case 3 is also not satisfied, the proposed method cannot be applied. One way to solve this problem is to increase the number of measurements until the condition defined by Eq. (41) is satisfied. In the remaining part of this section, a stochastic optimal guided missile control system[10] as shown in Fig. 2 is used for illustration. The parameters of the system are listed in the Appendix.

Example 1

Let the target acceleration A_t be a randomly reversing Poisson square wave with root-mean-square value β(ft/s^2), as shown in Fig. 3. If the number of the acceleration crossing zero is ν (times per second), then this target acceleration can be modeled[10] as the output of a first-order low-pass filter driven by a white noise w_t with power spectral density $\beta^2/2\pi\nu$ [(ft/s^2)^2/rad/s] $(-\infty < w < \infty)$. The correlation time constant of this shaping filter is $(2\nu)^{-1}$ seconds, as shown in Fig. 4. The state and the measurement equations

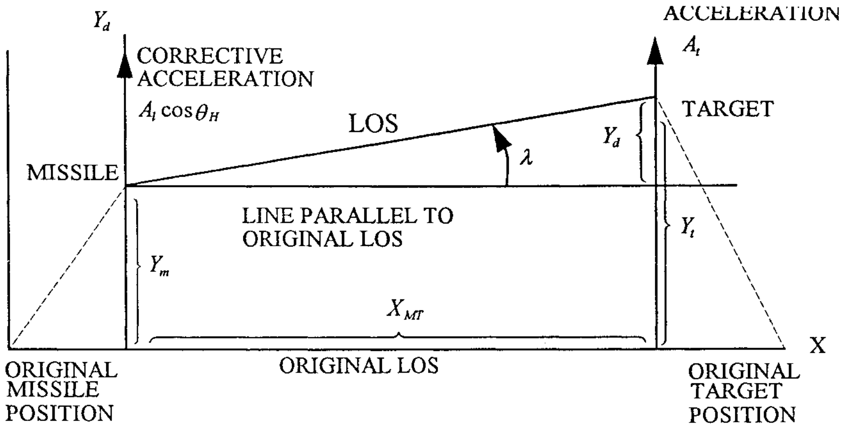


Fig. 2 Intercept geometry of guided missile system.

are given, respectively, as

$$\begin{bmatrix} \dot{y}_d \\ \ddot{y}_d \\ \dot{A}_t \end{bmatrix} = Ax + Bu + Fw$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2\nu \end{bmatrix} \begin{bmatrix} Y_d \\ \dot{Y}_d \\ A_t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} u$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\nu \end{bmatrix} w \tag{57}$$

and

$$y = Cx + v_2$$

$$= [1 \quad 0 \quad 0] \begin{bmatrix} Y_d \\ \dot{Y}_d \\ A_t \end{bmatrix} + v_2 \tag{58}$$

where Y_d is the lateral (miss) distance of missile and target, u is the missile lateral acceleration, and v_2 is a colored noise in this example, e.g., the glint noise, which is defined[1] as

$$\dot{v}_2 = -10v_2 + 10w_g \tag{59}$$

where w_g is a white noise with zero mean and variance R_2. According to Eqs. (3), (6–13), and (57–59), one has

$$D_a = 1 \tag{60}$$

$$v_a = v_2 \tag{61}$$

$$W_a = -10 \tag{62}$$

$$N_a = 10w_g \tag{63}$$

$$\Sigma = 100R_2 \tag{64}$$

and

$$L = \dot{D}_a + D_a W_a$$

$$= -10 \tag{65}$$

Therefore, case 1 of the proposed method can be applied with the new measurement equation defined by Eqs. (23–25) as

$$z = L_1\dot{y} - L_2 y - L_1 CBu$$

$$= H_1 x + v_{\text{white}} \tag{66}$$

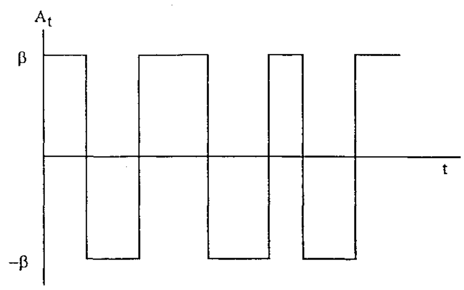


Fig. 3 Poisson square wave model of target maneuver.

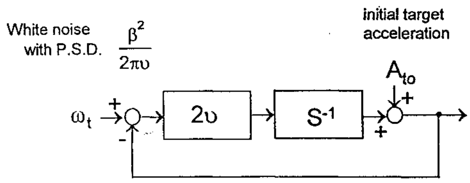


Fig. 4 Shaping filter model of target maneuver.

where

$$L_1 = D_a L^{-1}$$
$$= -0.1 \tag{67}$$

$$L_2 = I \tag{68}$$

$$H_1 = L_1(\dot{C} + CA) - C$$
$$= [-1.0 \quad -0.1 \quad 0] \tag{69}$$

$$v_{\text{white}} = L_1 CFw + L_1 D_a N_a$$
$$= -w_g \tag{70}$$

The covariance matrix Σ_1 and cross-variance matrix C_1 defined by Eqs. (26) and (33) are obtained as

$$\Sigma = L_1 CFQ(L_1 CF)^T + L_1 D_a \Sigma (L_1 D_a)^T$$

$$= R_2 \tag{71}$$

and

$$C_1 = Q(L_1 CF)^T$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{72}$$

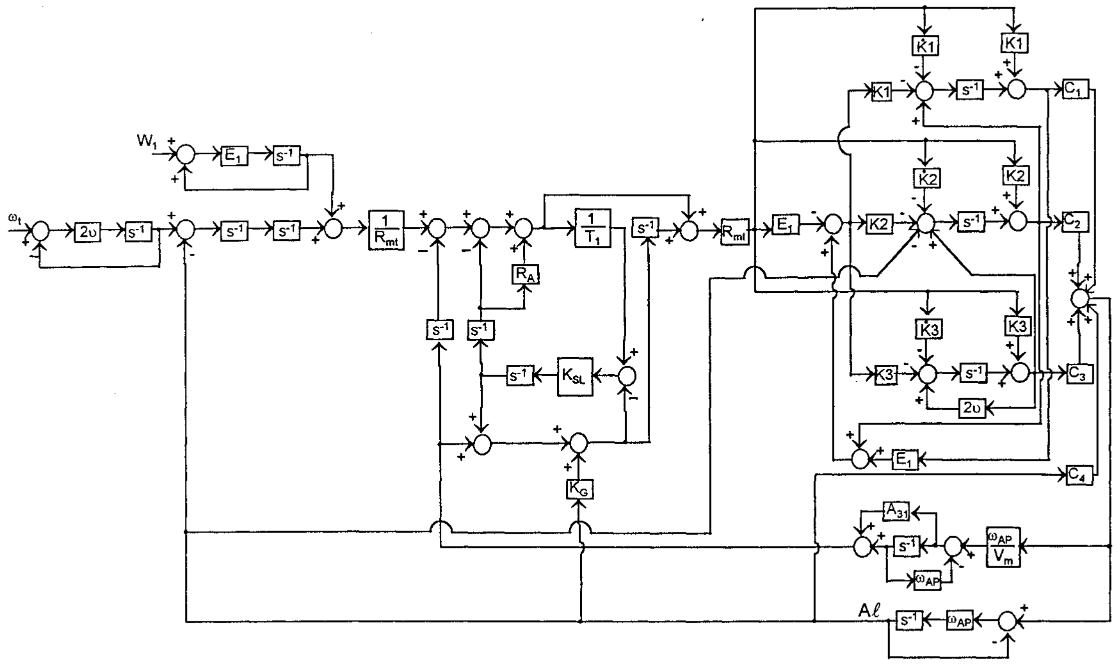


Fig. 5 Block diagram of whole guidance and control system with only glint noise.

By Eq. (34) the Kalman gain matrix is

$$K = \left(PH_1^T + BC_1\right)\Sigma_1^{-1} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \tag{73}$$

where

$$k_1 = (-1.0p_{11} - 0.1p_{12})R_2^{-1} \tag{74}$$

$$k_2 = (-1.0p_{12} - 0.1p_{22})R_2^{-1} \tag{75}$$

$$k_3 = (-1.0p_{13} - 0.1p_{23})R_2^{-1} \tag{76}$$

where the p_{ij} are defined by Eq. (35) as

$$\dot{p}_{11} = 2p_{12} - k_1^2 R_2 \tag{77}$$

$$\dot{p}_{12} = \dot{p}_{21} = p_{22} + p_{13} - k_1 k_2 R_2 \tag{78}$$

$$\dot{p}_{13} = \dot{p}_{31} = p_{23} - 2\nu p_{13} - k_1 k_3 R_2 \tag{79}$$

$$\dot{p}_{22} = 2p_{23} - k_2^2 R_2 \tag{80}$$

$$\dot{p}_{23} = \dot{p}_{32} = p_{33} - 2\nu p_{23} - k_2 k_3 R_2 \tag{81}$$

$$\dot{p}_{33} = -4\nu p_{33} + 4\nu\beta^2 - k_3^2 R_2 \tag{82}$$

The block diagram of the whole guidance and control system with a typical seeker is shown in Fig. 5. The optimal control gains C_i' in Fig. 5 can be found in Ref. 11. The simulation results of Kalman gains are shown in Fig. 6. By the adjoint simulation method,[1] the miss distances due to target maneuver and the glint noise are as shown in Fig. 7a. It can be seen that the glint noise plays an important role near interception.

It should be noted that if the glint noise were a white noise, then, by the proposed method, one can model the glint noise as a colored one with the corresponding shaping filter bandwidth set to be 10 times that of the system fastest mode (i.e., the stabilization loop bandwidth of the seeker $K_{SL} = 100$ rad/s, as shown in Fig. 5). Thus

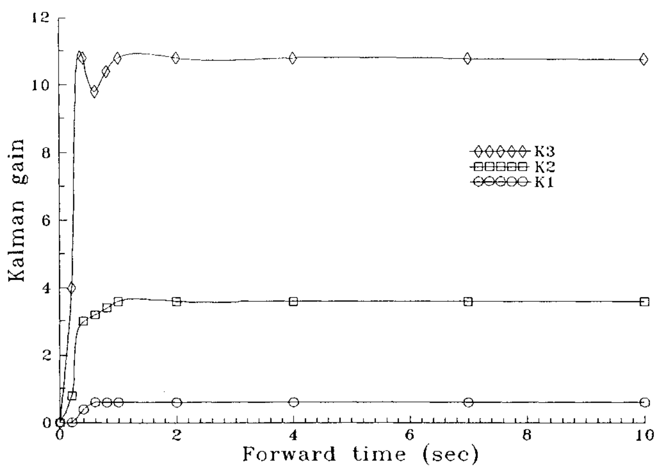


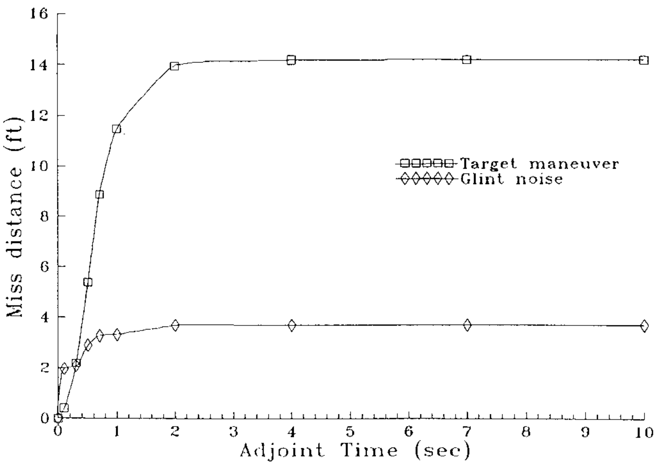
Fig. 6 Kalman gain for system with glint (colored) noise.

$B_w = 1000$ rad/s in this example, and it is 100 times larger than that of the original true glint noise. The relationship[10] of the bandwidth B_w, power spectral density Φ_g, and root-mean-square value σ_{rms} of the glint noise is $\Phi_g = 2\sigma_{rms}^2/B_w$. Therefore, in order to obtain the same root-mean-square value of the original glint noise (which is often on the order of a fifth of the wingspan[12]), the new power spectral density of the glint noise would be reduced by 100 times. Therefore, the miss distance due to the white glint noise would be reduced by 10 times the original colored glint noise, which is shown in Fig. 7b for comparison. This is the reason why the frequency agility technique[12] is often used to reduce the miss distance due to glint noise.

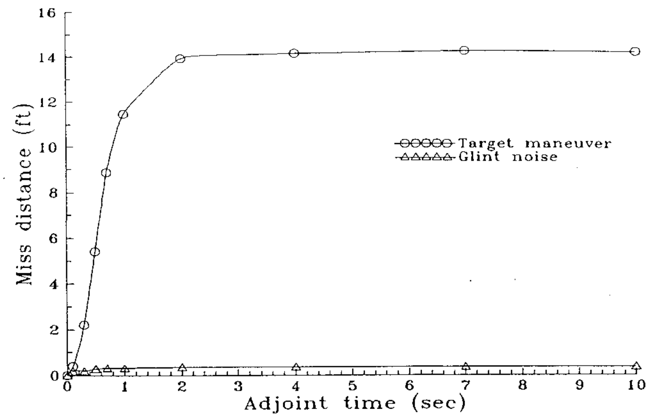
Example 2

Let the measurement equation in Example 1 be

$$y = Cx + Dv + Ed_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{(t_f-t)^2 V_c^2}{R_0} \\ \frac{10(t_f-t)^2 V_c^2}{R_0} \end{bmatrix} v + \begin{bmatrix} (t_f - t)V_c \\ 100(t_f - t)V_c \end{bmatrix} d_w \tag{83}$$



a)



b)

Fig. 7 Miss distance due to target maneuver and a) glint colored noise and b) glint (white) noise.

where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{84}$$

$$D = \begin{bmatrix} \dfrac{(t_f - t)^2 V_c^2}{R_0} \\ \dfrac{10(t_f - t)^2 V_c^2}{R_0} \end{bmatrix} = \begin{bmatrix} d_{11} \\ d_{21} \end{bmatrix} \tag{85}$$

$$E = \begin{bmatrix} (t_f - t)V_c \\ 100(t_f - t)V_c \end{bmatrix} = \begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix} \tag{86}$$

where d_w is the white fading noise with zero mean and variance J_1, t_f is the total time of engagement for the homing range R_0, V_c is the closing velocity of missile and target, and v is now a receiver (colored) noise, defined as

$$\dot{v} = gv + a \tag{87}$$

where a is a white-noise vector with zero mean and covariance R.

It should be noted that since there are both colored and white measurement noises in this example the original measurement differentiation method cannot be applied. Then one should apply the state augmentation method by defining the augmented state as

$$x_a = \begin{bmatrix} x \\ v \end{bmatrix} \tag{88}$$

By Eqs. (57), (87), and (88),

$$\dot{x}_a = A_a x_a + B_a u + F_a w_a \tag{89}$$

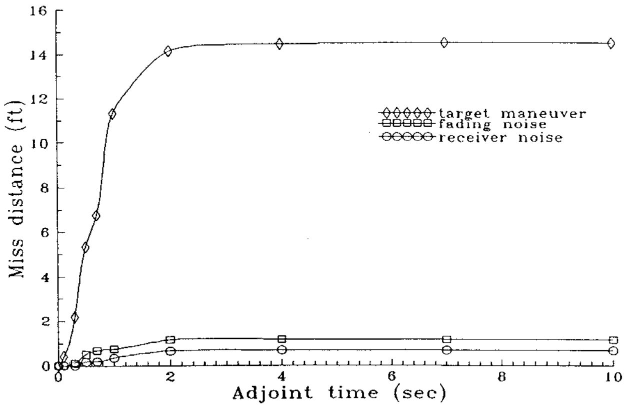


Fig. 8 Miss distance due to target maneuver, receiver, and fading noises.

where

$$A_a = \begin{bmatrix} A & 0 \\ 0 & g \end{bmatrix} \tag{90}$$

$$B_a = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{91}$$

$$F_a = \begin{bmatrix} F & 0 \\ 0 & 1 \end{bmatrix} \tag{92}$$

$$w_a = \begin{bmatrix} w \\ a \end{bmatrix} \tag{93}$$

with A, B, and F to be defined in Eq. (57), and Eq. (83) becomes

$$y = C_a x_a + E d_w \tag{94}$$

where

$$C_a = [C \quad D] \tag{95}$$

with C, D, and E defined from Eqs. (84–86). Then the standard Kalman filter estimation algorithm can be applied according to the problem defined from Eqs. (89–95). Since the order of the new state equation is increased, the loading of the estimator is also increased. However, this problem can be avoided by the proposed method as defined in Eqs. (4) and (5) to approximate and replace the white fading noise d_w by a colored noise b, i.e.,

$$d_w \cong b \tag{96}$$

and b is defined as

$$\dot{b} = -B_w b + B_w f, \quad B_w = 10 w_0 \tag{97}$$

where w_0 is the bandwidth of the system fastest mode (i.e., the stabilization loop bandwidth of the seeker, K_{SL} = 100 rad/s, as shown in Fig. 5), e.g., w_0 = 100 rad/s and f is a white-noise vector with zero mean and covariance J_f to be defined as $2J_1/B_w$. Then one can augment this new colored noise b with the original colored measurement noises and yield

$$y = Cx + D_a v_a = Cx + \begin{bmatrix} d_{11} & e_{11} \\ d_{21} & e_{21} \end{bmatrix} \begin{bmatrix} v \\ b \end{bmatrix} \tag{98}$$

where

$$D_a = \begin{bmatrix} d_{11} & e_{11} \\ d_{21} & e_{21} \end{bmatrix} \tag{99}$$

and the augment colored noise v_a is

$$v_a = \begin{bmatrix} v \\ b \end{bmatrix} \tag{100}$$

which is defined as

$$\begin{aligned} \dot{v}_a &= \begin{bmatrix} \dot{v} \\ \dot{b} \end{bmatrix} \\ &= \begin{bmatrix} g & 0 \\ 0 & -10w_0 \end{bmatrix} \begin{bmatrix} v \\ b \end{bmatrix} + \begin{bmatrix} a \\ 10w_0 f \end{bmatrix} \\ &= W_a v_a + N_a \end{aligned} \tag{101}$$

where

$$W_a = \begin{bmatrix} g & 0 \\ 0 & -10w_0 \end{bmatrix} \tag{102}$$

$$N_a = \begin{bmatrix} a \\ 10w_0 f \end{bmatrix} \tag{103}$$

Then by Eqs. (13), (99), and (102), one has

$$\begin{aligned} L &= \dot{D}_a + D_a W_a \\ &= \begin{bmatrix} \dot{d}_{11} + d_{11}g & \dot{e}_{11} - 10w_0 e_{11} \\ \dot{d}_{21} + d_{21}g & \dot{e}_{21} - 10w_0 e_{21} \end{bmatrix} \end{aligned} \tag{104}$$

which is nonsingular; therefore, one can apply case 1 of the proposed method by making a new measurement equation defined from Eqs. (23–25), i.e.,

$$\begin{aligned} z &= L_1 \dot{y} - L_2 y - L_1 C B u \\ &= H_1 x + v_{\text{white}} \end{aligned} \tag{105}$$

where

$$H_1 = L_1(\dot{C} + CA) - C \tag{106}$$

$$v_{\text{white}} = L_1 C F w + L_1 D_a N_a \tag{107}$$

with the related parameters defined from Eqs. (83–104). The other formulations are similar to those derived from Eqs. (33–40) and shown in Fig. 1. The miss distances are shown in Fig. 8. Comparing Figs. 7a and 8, one can see that the miss distances due to both the receiver (colored) noise and the fading (white) noise are not as critical as that due to the glint (colored) noise, which is consistent with results obtained previously.[1,2,12]

IV. Conclusions

1) In this paper a modified measurement differentiation method is proposed for the time-varying stochastic control systems with both white and colored measurement noises.

2) The key point of this method is to approximate the white noise by a wide-band colored one and model the latter by a first-order shaping filter, set the bandwidth to 10 times that of the system fastest mode, and then augment these new colored noises with the original colored ones.

3) However, if the number of measurement outputs is less than that of the augmented colored noise, the proposed method cannot be directly applied. The way to solve this problem is to increase the number of measurements.

4) By this the proposed method can take both colored and white measurement noises into consideration without increasing the order of the state equation. This advantage cannot be obtained by the measurement differentiation method or the state augmentation method.

5) Therefore, the proposed method is simpler, general, and practical. A guided missile stochastic optimal control system with time-varying colored and white measurement noises is also given for illustration.

V. Appendix

1. $\beta = 160$ ft/s^2
2. $\nu = 0.1$ s^{-1}
3. $w_{\text{AP}} = 10$ rad/s
4. $R_0 = 20{,}000$ ft
5. $V_c = 5$ Mach
6. $R_2 = 4$ ft^2
7. $J_1 = 1.6 \times 10^{-7}$ rad^2
8. $T_1 = 0.1$ s
9. $R_A = 0$
10. $K_{\text{SL}} = 100$ rad/s
11. $K_G = 0$ rad/s/g
12. $A_{31} = 0.25$ s
13. $R = 0.8$ rad^2
14. $g = -100$

Acknowledgment

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